Analytical Performance Evaluation of Manufacturing Cells Using Petri Net Model and Moment Generating Function

Yong-Soo Park and Jongwon Kim
Dept. of Mechanical Design and Production Engineering
Seoul National University
Kwanak-ku, Seoul 151-742, Korea

ABSTRACT

In this paper, a manufacturing cell is represented as a generalized stochastic Petri net model. Then, by applying moment generating function based approach, the steady-state probability formulation, and accordingly, the production rate and the work-in-process of the manufacturing cell are expressed, respectively, as a symbolic function of all the timed transition rates, such as the part processing rate, the failure rate and the repair rate. Finally, the analytical results are compared with the previous research results to verify the validity of the suggested method.

1. INTRODUCTION

In the design and operation of a manufacturing system, performance modeling and evaluation play an important role. Performance evaluation involves computing performance measures such as the production rate, the work-in-process, the part production lead time and the utilization rate of the entire system.

The general analytical tools used in the performance modeling and evaluation of the manufacturing system are queueing networks [1], Markov chains [2], and Petri net (PN) modeling [3]. Among them, Generalized Stochastic Petri net (GSPN) approach provides an effective and convenient framework to model manufacturing systems and for conducting performance studies, especially, when the concurrency, asynchronous events, logical precedence relations, and structural interactions within the manufacturing system should be modeled easily and efficiently [4, 5, 6].

However, previous works on the performance evaluation of manufacturing systems [6, 7] rely on software packages, such as SPNP (Stochastic Petri net Package) or SPSS (Stochastic Petri net Steady-state Solver) to solve the steady-state probability balance equations. If any parameter of the manufacturing system is altered, the steady-state probability equation has to be corrected and solved again. Therefore, closed-form solutions on the performance measures can not be obtained.

The main objective of this paper is to develop an analytical method for obtaining the closed-form symbolic performance evaluation functions of the transfer line. The production rate and the work-in-process of the transfer line are expressed, respectively, as a closed-form symbolic function of all the timed transition rates, such as the part processing rate, the failure rate and the repair rate. Guo *et al.* [8] presented an approach based on moment generating function (MGF) for performance analysis and has applied it to a simple machine-repairman Petri net model to obtain a closed-form symbolic performance evaluation function. In this paper, Guo's approach is extended to the case of a transfer line.

The MGF-based approach is briefly reviewed in the next section. In Section 3, the MGF-based approach is applied to evaluate a 2-machine 1-buffer (2M1B) transfer line. Two machines are assumed to be unreliable, and the amount of parts in buffer storage is assumed to be finite. Section 4 presents the analytical results on the performance evaluation and its comparison with the previous works of Gershwin and Berman [2] and Al-jaar and Desrochers [6]. Finally, summaries and conclusions are presented.

2. MOMENT GENERATING FUNCTION (MGF) BASED APPROACH

In this section, the MGF-based approach is briefly introduced. More details can be found in Guo *et al.* [8]. Firstly, the manufacturing system is modeled as a Petri net. Especially, in this paper, all the manufacturing systems are modeled by using Generalized Stochastic Petri net (GSPN).

A GSPN is basically a stochastic Petri net (SPN) with transitions that are either timed or immediate with the following definition :

A GSPN is a five-tuple (P, T, A, M_0, F), where (1) $P = \{p_1, p_2, \cdots, p_n\}$ is a set of places, (2) $T = \{t_1, t_2, \cdots, t_m\}$ is a set

0-7803-3280-6/96/$5.00

of transitions, (3) $A \subseteq (P \times T) \cup (T \times P)$ is a set of arcs, (4) $M_0 : P \rightarrow N$ is a mapping called initial marking that associates zero or more tokens to each place, (5) with $m, n \geq 0$, $m + n \geq 1$, and $P \cap T = 0$, and additionally, (6) T is partitioned into two sets: T_I of immediate transitions and T_E of exponential transitions, (7) $F : R[M_0] \times T_E \rightarrow R$ is a firing function that associates with each $t \in T_E$ in each reachable marking $M \in R[M_0]$, a real number $F(M,t)$, and (8) each $t \in T_I$ has zero firing time and each $t \in T_E$ has an exponentially distributed firing time with a real number $F(M,t)$ in each $M \in R[M_0]$.

In the graphical representation of a GSPN, a horizontal line represents an immediate transition and a rectangular thick bar represents an exponential transition. GSPN markings are of two types: those in which only exponential transitions are enabled are designated as *tangible* markings while the rest of the markings are called *vanishing* markings. In a tangible marking, any of the enabled exponential transitions can fire next. The probability that a given transition fires next depends upon the firing rates of the enabled exponential transitions. In a vanishing marking, the firing rules are as follows:

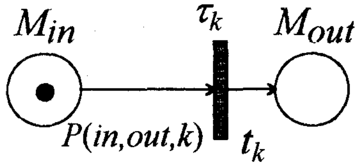
(1) Only the enabled immediate transitions are allowed to fire. (2) If two or more concurrent immediate transitions are enabled, all of them fire simultaneously. (3) If some of the enabled immediate transitions are conflicting, only one of them is allowed to fire at a time, according to a predefined probability distribution. Such distributions are called random switches.

The second step of the MGF-based approach is to construct the reachability graph of the obtained GSPN model of the manufacturing system, and then, to transform the reachability graph into a *state machine Petri net (PN)*. The state machine PN is a special class of PN, which has following properties:

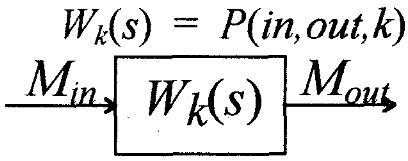
(1) Each transition has one and only one input and output places. (2) There is only one token in the net. In summary, the second step yields a new state machine PN, which actually represents the reachability graph. Hence, each marking M_i (i = 0, 1, 2,···, N, where N is the number of all $R[M_0]$) in the reachability graph corresponds to a place of the new state machine PN.

In the third step of the MGF-based approach, the new state machine PN, which has been generated in the previous step, is converted into the transfer function form by using the following definition:

A state machine PN, with only one input place M_{in}, only one output place M_{out}, and an exponential transition t_k whose firing time T_k is exponentially distributed with the firing rate τ_k [see Fig.1.a], is supposed to be equivalent to the transfer function [see Fig.1.b] of



(a) A state machine Petri net



(b) An equivalent transfer function

Fig.1. Definition of a transfer function

$$W_k(s) = P(in,out,k)\, G(s) \tag{1}$$

where $P(in,out,k)$ is defined to be the probability that transition t_k from M_{in} to M_{out} can fire, and $G(s)$ is the moment generating function (MGF) defined as follows :

$$G(s) = \int_{-\infty}^{\infty} e^{sT} f(T)\,dT \text{ with } f(T) = \begin{cases} 0 & T \leq 0 \\ \tau_k e^{-\tau_k T} & T > 0 \end{cases} \tag{2}$$

or
$$G(s) = \frac{\tau_k}{\tau_k - s} \tag{3}$$

where s is an extended parameter, and $f(T)$ is the probability density function of the exponential random variable T with the rate of τ_k. The transition probability $P(in,out,k)$ depends upon τ_k and the time rates of other enabled transitions at that marking.

Once the transfer function form is obtained, an equivalent signal flow graph can be obtained. A place in a state machine PN corresponds to a node in the signal flow graph, and the transfer function attached to a transition corresponds to the gain. Then, any equivalent transfer function $W_E^{ij}(s)$ between two nodes (that is, the places of the state machine PN, which represent markings in the reachability graph) M_i and M_j $(i, j = 0,1,2, \cdots, N)$ can be generated through the reduction technique by using Mason's rule [9]. Finally, the steady-state probability of each marking M_j (j = 0,1,2, ···, N) are calculated as follows:

(1) The steady-state probability of a place M_i of the state machine PN equals the ratio of its *mean sojourn time* to its *mean recurrence time*.
(2) The mean recurrence time T_{ii} of a place M_i in the state machine PN is the expected total time of regeneration. In order to find the mean recurrence time, the state machine

PN loop is split at the node M_i by introducing a new virtual place M_i^* and directing all input transitions of M_i in the loop to M_i^*. Then, the equivalent transfer function $W_E^{ii}(s)$ from M_i and M_i^* can be obtained by using Mason's rule. From equation (1), since the transition probability $P(i,i,k)$ is always equal to 1, the equivalent MGF $G_{ii}(s)$ is equal to $W_E^{ii}(s)$. Therefore, since the mean recurrence time is equal to the first moment of the equivalent MGF,

$$T_{ii} = \frac{\partial}{\partial s} W_E^{ii}(s) \Big|_{s=0} \quad (4)$$

(3) The mean sojourn time T_j of a place M_j can be calculated using the same way to calculate the recurrence time T_{ii}. However, in this case, the mean sojourn time T_j is calculated supposing that it is a special mean recurrence time T_{ii}^* in the sense that only the output transitions of the place M_j are timed transitions and that the output transitions of the other places in the Petri net are assumed to be immediate transitions. Nevertheless, the branch probabilities of these immediate transitions should be retained. Accordingly, a new equivalent transfer function $W_E^{ii}(s)^*$ should be obtained first by modifying $W_E^{ii}(s)$ in step (2). Then, the mean sojourn time T_j of a place M_j is obtained as follows. For details, refer to the case studies presented in the next sections.

$$T_j = \frac{\partial}{\partial s} W_E^{ii}(s)^* \Big|_{s=0} \quad (5)$$

(4) Finally, the steady-state probability π_j of a place M_j is calculated as follows :

$$\pi_j = \frac{T_j}{T_{ii}} \quad (j = 0,1,2,\cdots,N) \text{ for any } i = 0,1,2,\cdots,N \quad (6)$$

3. APPLICATION TO A TRANSFER LINE

Gershwin and Berman [2] have derived a closed-form solution for the analysis of a serial transfer line consisting of two unreliable machines with random processing times and a finite buffer (2M1B line, see Fig.2). Their approach was based on Markov chain modeling. Al-jaar and Desrochers [6] used the GSPN basic modules for machines and buffers as building blocks to model and analyze serial transfer lines. They have compared the performance evaluation results of their GSPN approach with those of Gershwin and Berman. Both results showed good agreement with each other. Hence, in this paper, the same 2M1B transfer line used in both approaches mentioned above is selected as the case study for demonstrating the MGF-based approach.

The transfer line consists of two machines that are separated by a finite storage buffer (Fig.2). Parts enter machine #1 from outside. Each part is processed in machine #1. Then it is passed to the buffer and proceeds to machine #2. After being processed in machine #2, the part leaves the line. The machine may be in one of three possible states: ***processing***, ***idle***, and ***under repair***. When the machine is processing a part, it is in the state '*processing*'. When the machine is ***starved*** or ***blocked***, it is in the state '*idle*'. It is called '***starved***', when the machine is waiting for a part to enter from the upstream machine. It is called '*blocked*', when there is no room in the buffer storage of the next machine to put the processed part. It is assumed here that the first machine is never starved and the second machine is never blocked. When the machine is out of order, it is in the state '***under repair***'. When the machine goes from *processing* to *under repair*, it is said to '*fail*'. A '***repair***' takes place when the transition from ***under repair*** to ***processing*** occurs. It is assumed here that the machine never fails if it is in the idle state.

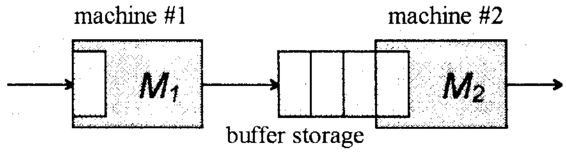


Fig.2. A two-machine one-buffer (2M1B) transfer line

It is also assumed that machine #1 does not process a part if the downstream buffer part storage is full. The number of ***buffer part storage*** is represented by ***n***. This is the sum of the number of parts in the buffer plus the number of part in machine #2. For example, the maximum possible value of ***n*** is 4 in Fig.2, since there is one part in machine #2 and three parts can be stored in the buffer.

Major assumptions made in the previous works [2, 6] are as follows:

(1) The part processing time of the machine is not constant; rather it is assumed to be exponentially distributed with the *processing rate* of μ [parts/unit-time]. (2) The failure time, that is, the time interval from the failure recovery to the next failure, is assumed to be exponentially distributed with the *failure rate* of f [number-of-failure /unit-time]. (3) The repair time, that is, the time interval from the failure moment to the failure recovery, is also assumed to be exponentially distributed with the *repair rate* of ***r*** [number-of-repair/unit-time].

Figure 3 represents the GSPN model of the 2M1B transfer line defined so far. The interpretations of the place p and the transition t are as follows:

$p(m,i)$: machine (m/c) #*m* is idle.
$p(m,p)$: m/c #*m* is processing a part.
$p(m,d)$: m/c #*m* is under repair.

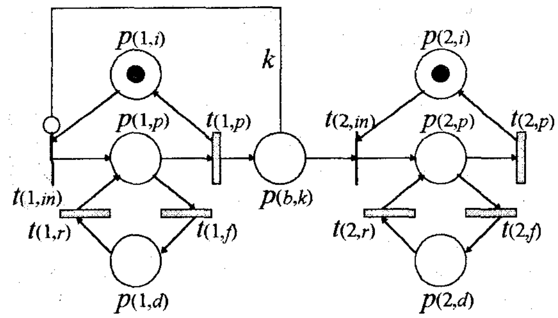


Fig.3. GSPN model of a 2M1B transfer machine

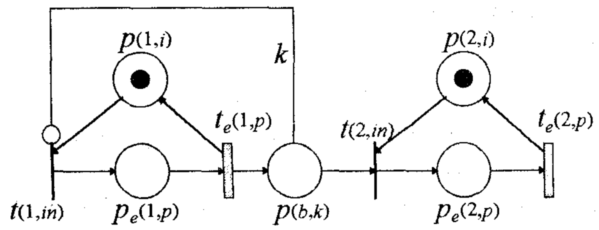


Fig.4. Reduced GSPN model of a 2M1B transfer machine

$p(b,k)$: part(s) is stored in the buffer of capacity $k = n$-1.
$t(m,in)$: part has been supplied to m/c #m immediately.
$t(m,p)$: m/c #m has processed a part with the rate $\mu(m)$.
$t(m,f)$: m/c #m has failed with the rate $f(m)$.
$t(m,r)$: m/c #m has been repaired with the rate $r(m)$.

The GSPN model of the 2M1B transfer line shown in Fig.3 can be further reduced by using the concept of "isolated efficiency" [2] or "throughput equivalent" [6]. The *isolated efficiency e(m)* of machine #m is defined as

$$e(m) = \frac{r(m)}{r(m) + f(m)} \tag{7}$$

This is the fraction of time that machine #m would be producing parts if it were isolated, that is, if it had an endless supply of raw parts and unlimited reservoir in which to store processed parts. Hence, the *isolated processing rate* of machine #m, $\mu_e(m)$, is given by

$$\mu_e(m) = \mu(m)e(m) \tag{8}$$

This is the rate at which machine #m would process parts in isolation. Hence, the reduced GSPN model of the 2M1B transfer line is obtained as shown in Fig.4. In this case, the interpretation of the newly defined place and transition are as follows:

$p_e(m,p)$: machine #m is processing a part.
$t_e(m,p)$: machine #m has processed a part with *the isolated processing rate* of $\mu_e(m)$.

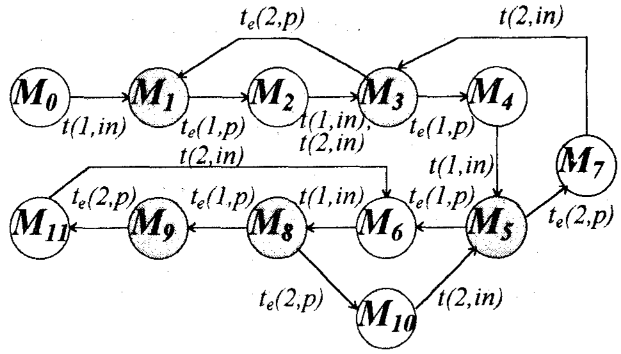


Fig.5. Reachability graph of the reduced GSPN model of a 2M1B transfer machine with n = 4.

As the next step, the reachability graph of the reduced GSPN model (with n = 4) in Fig.4 is constructed as shown in Fig.5. The tangible markings are M_1, M_3 , M_5 , M_8 and M_9 represented by dark ovals. The other markings are vanishing markings, which are represented by white ovals. Here the marking M_j is defined as follows:

$$M_j = \begin{bmatrix} M_j\{p(1,i)\} \\ M_j\{p_e(1,p)\} \\ M_j\{p(b,k)\} \\ M_j\{p(2,i)\} \\ M_j\{p_e(2,p)\} \end{bmatrix}^T \tag{9}$$

The initial marking M_0 is [1,0,0,1,0], which means that both machines #1 and #2 are in the idle state. The other markings are defined as follows:

M_1 = [0,1,0,1,0]; M_2 = [1,0,1,1,0]; M_3 = [0,1,0,0,1];
M_4 = [1,0,1,0,1]; M_5 = [0,1,1,0,1]; M_6 = [1,0,2,0,1];
M_7 = [0,1,1,1,0]; M_8 = [0,1,2,0,1]; M_9 = [1,0,3,0,1];
M_{10} = [0,1,2,1,0]; M_{11} = [1,0,3,1,0];

Now, the vanishing markings are removed from the reachability graph in Fig.5, and then, the reachability graph of the tangible markings is transformed to the state machine PN shown in Fig.6. As the next step, the new state machine PN is converted into the transfer function form by using the MGF definitions (1), (2) and (3) in the previous section. Once the transfer function form is obtained, an equivalent signal flow graph can be obtained. The place, that is the marking, in the state machine PN corresponds to the node in the signal flow graph, and the transfer function corresponds to the gain.

Then, the equivalent transfer function $W_E^{11}(s)$ from M_1 and M_1* can be obtained by using Mason's rule as follows:

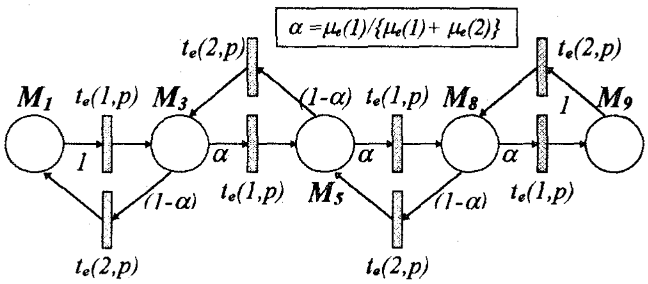


Fig.6. State machine Petri net in case of the 2M1B transfer machine (n = 4)

$$W_E^{11}(s) = \frac{\mu_e(1)\mu_e(2)[s^3 - As^2 + Bs - \mu_e(2)^3]}{(s-\mu_e(1))[s^4 - Cs^3 + Ds + \mu_e(2)^4]} \tag{10}$$

where, $\mu_e(m) = \dfrac{\mu(m)r(m)}{r(m) + f(m)}$; $m = 1, 2$

$$A = 2\mu_e(1) + 3\mu_e(2)$$
$$B = [\mu_e(1) + \mu_e(2)]^2 + 2\mu_e(2)^2$$
$$C = 3\mu_e(1) + 4\mu_e(2)$$
$$D = [\mu_e(1) + \mu_e(2)]^3 + \mu_e(2)[3\mu_e(2)^2 - \mu_e(1)^2]$$

Hence the mean recurrence time is calculated as follows:

$$T_{11} = \frac{\mu_e(1)^4 + \mu_e(1)^3\mu_e(2) + \mu_e(1)^2\mu_e(2)^2 + \mu_e(1)\mu_e(2)^3 + \mu_e(2)^4}{\mu_e(1)\mu_e(2)^4} \tag{11}$$

The mean sojourn time T_j of a place M_j (j = 1,3,5,8,9) can be calculated supposing that it is a special mean recurrence time T_{ii}* in the sense that only the output transitions of the place M_j are timed transitions and that the output transitions of the other places in the Petri net are assumed to be immediate transitions. Nevertheless, the branch probabilities of these immediate transitions should be retained. In this case, the mean sojourn time T_j of a place M_j (j = 1,3,5,8,9) are calculated as follows:

$$T_1 = \frac{1}{\mu_e(1)} \qquad T_3 = \frac{1}{\mu_e(2)} \qquad T_5 = \frac{\mu_e(1)}{\mu_e(2)^2}$$

$$T_8 = \frac{\mu_e(1)^2}{\mu_e(2)^3} \qquad T_9 = \frac{\mu_e(1)^3}{\mu_e(2)^4} \tag{12}$$

Finally, the steady-state probability π_j of a place M_j is obtained as follows :

$$\pi_j = \frac{T_j}{T_{11}} \quad (j = 1,3,5,8,9) \tag{13}$$

4. ANALYTICAL PERFORMANCE EVALUATION

Performance measures are obtained from the steady-state probabilities of markings. Using these probabilities, it is possible to compute the expected number of tokens in each place, the probability of a place to have a certain number of tokens, and the probability that a transition is enabled. From these, the performance measures such as the average production rate, the average work-in-process, the average production lead time and the average utilization rate of the transfer line can be obtained.

For example, the average production rate P is obtained from the following product form:

$$P = \mu_e(2) \cdot prob[t_e(2, p) \text{ being enabled}]$$

where $\mu_e(2)$ is the firing rate of the transition $t_e(2, p)$, and '$prob[t_e(2, p)$ being enabled]' refers to the summation of the steady state probabilities that the transition $t_e(2, p)$ is enabled. Hence, the explicit closed-form of the average production rate P is derived as follows:

$$P = \mu_e(2)\,[\pi_3 + \pi_5 + \pi_8 + \pi_9] = \frac{\mu_e(1)\mu_e(2)[\mu_e(1)^3 + \mu_e(1)^2\mu_e(2) + \mu_e(1)\mu_e(2)^2 + \mu_e(2)^3]}{\mu_e(1)^4 + \mu_e(1)^3\mu_e(2) + \mu_e(1)^2\mu_e(2)^2 + \mu_e(1)\mu_e(2)^3 + \mu_e(2)^4} \tag{14}$$

The average work-in-process N is confined to the number of parts staying in the buffer storage and machine #2 in order to compare the result with that of Gershwin's [2]. The explicit closed-form solution of the average work-in-process N is derived as follows:

$$N = 1\cdot\pi_3 + 2\cdot\pi_5 + 3\cdot\pi_8 + 4\cdot\pi_9 = \frac{\mu_e(1)[4\mu_e(1)^3 + 3\mu_e(1)^2\mu_e(2) + 2\mu_e(1)\mu_e(2)^2 + \mu_e(2)^3]}{\mu_e(1)^4 + \mu_e(1)^3\mu_e(2) + \mu_e(1)^2\mu_e(2)^2 + \mu_e(1)\mu_e(2)^3 + \mu_e(2)^4} \tag{15}$$

The performance evaluation results are presented as to the following specific cases:

CASE 1	$0.1 \le \mu(1) \le 1000$	refer to Fig.7,
CASE 2	$0.1 \le \mu(2) \le 1000$	refer to Fig.7,
CASE 3	$0.1 \le f(1) \le 1000$	refer to Fig.8,
CASE 4	$0.1 \le f(2) \le 1000$	refer to Fig.8

The results of CASE 1 to CASE 4 were obtained using k = 3 (i.e. n = 4). The other standard parameter values were $\mu(1)$ = 1, $\mu(2)$ = 2, $f(1)$ = 3, $f(2)$ = 4, $r(1)$ = 5, $r(2)$ = 6. Here the unit-time is assumed to be *an hour*. Figure 7 shows that the production rate increases to a limit as the processing rates increases. *P(i)* and *N(i)* is, respectively, the production rate and the work-in-process of the system in CASE i.

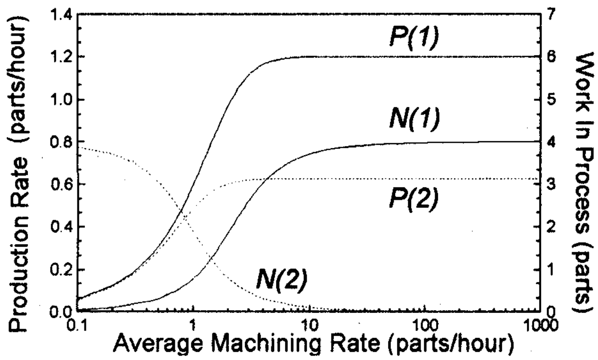


Fig.7. Effects of the machining rate on the production rate and work-in-process

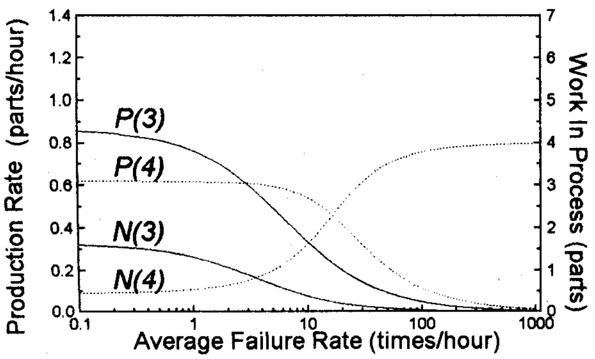


Fig.8. Effects of the failure rate on the production rate and work-in-process

In Fig.8, failure rates are varied. System production rates and average work-in-process are plotted. As failure rate increases, production rates decreases.

Comparing these results with those obtained using the closed-form solution by Gershwin and Berman [2] and those obtained using a GSPN model and a stochastic PN software package [6] shows that the MGF-based approach duplicates the results obtained by direct MC modeling and stochastic PN analysis. The main advantages of the MGF-based approach are that explicit symbolic closed-form expressions for performance measures can be obtained. Even the closed-form solution by Gershwin and Berman [2] requires the iterative solving of the cubic equation and a series of linear equations for obtaining each set of parameters for each case.

5. CONCLUSIONS

By applying the MGF-based approach a 2M1B transfer line, the main advantages of the MGF-based approach are demonstrated. It is possible to obtain explicit closed-form symbolic expression functions of the performance measures (such as the production rate and the work-in-process) for the variations of the production rates, failure rates, and repair rates of the machines. It is also verified that the derived explicit closed-form symbolic expression functions of the performance measures duplicate the results obtained by direct MC modeling and stochastic PN analysis.

ACKNOWLEDGMENT

This work is partly supported by the Engineering Research Center for Advanced Control and Instrumentation (ERC-ACI) at Seoul National University in Korea.

REFERENCES

[1] Tempelmeier, H. and H. Kuhn, *Flexible Manufacturing Systems*, John Wiley & Sons, 1993.

[2] Gershwin, S. B. and O. Berman, "Analysis of Transfer Lines Consisting of Two Unreliable Machines with Random Processing Times and Finite Storage Buffers," AIIE Trans., Vol.13, No.1, 1981, pp.2-11.

[3] Viswanadham, N. and Y. Narahari, *Performance Modeling of Automated Manufacturing Systems*, Prentice Hall, 1992.

[4] Balbo G., G. Chiola, G. Franceschinis and G. M. Roet, "Generalized Stochastic Petri Nets for The Performance Evaluation of FMS," *Proc. of IEEE Int. Conf. on Robotics and Automat.*,1987, pp.1013-1018.

[5] Narahari, Y. and N. Viswanadham, "A Petri Net Approach to The Modelling and Analysis of Flexible Manufacturing Systems," *Annals of Operations Research*, Vol.3, pp.449-472, 1985.

[6] Al-Jaar, R. Y. and A. A. Desrochers, "Performance Evaluation of Automated Manufacturing Systems Using Generalized Stochastic Petri Nets," *IEEE Trans. on Robotics and Automat.*, Vol.6, No.6, 1990, pp.621-639.

[7] Viswanadham, N. and Y. Narahari, "Stochastic Petri Net Models for Performance Evaluation of Automated Manufacturing Systems," *Information and Decision Technologies*, Vol.14, 1988, pp. 125-142.

[8] Guo, D., F. DiCesare and M. Zhou, "A Moment Generating Function Based Approach for Evaluation Extended Stochastic Petri Nets," *IEEE Trans. on Automatic Control*, Vol.38, No.2,1993, pp.321-327.

[9] Zhou, M. C., C. Wang and X. Zhao, "Automating Mason's Rule and Its Application to Analysis of Stochastic Petri Nets," *IEEE Trans. on Control Systems Technology*, Vol.3, No.2, 1995, pp.238-244.